

**Geometry for elementary school**

# **Print version**

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## Feedback

Feedback is very important in many topics, especially when writing a book like this. We would like to learn from your experience using this book. What was the age of the child that used the book? What did he like most? Which topics did he have problem understanding? What ideas for improvements do you have?

## Help

## Questions & Answers

Have a question? Why not ask the very textbook that you are learning from?

## Further Reading

- Geometry

## External Links

- Euclid's Elements (<http://aleph0.clarku.edu/~djoyce/java/elements/toc.html>)

# Introduction

<b>Geometry for elementary school</b>		
	<b>Introduction</b>	<b>Our tools: Ruler and compass</b>

## Why geometry?

Geometry is one of the most elegant fields in mathematics. It deals with visual shapes that we know from everyday life, yet uses accurate proofs.

Learning geometry does not require previous skills like basic arithmetic. Hence, geometry is suitable as an introduction to mathematics for elementary school.

## Who should use this book?

This book is intended for use by a parent (or a teacher) and a child. It is recommended that the parent will be a bit familiar with geometry but this is not necessary. The parent can simply read the chapter before teaching the child and then learn it with it.

## Book guidelines

The classic book about geometry is Euclid's Elements (<http://aleph0.clarku.edu/~djoyce/java/elements/toc.html>) . This book helped teaching geometry for hundreds of years, so we feel that writing this book based on the elements is a correct step.

We will adapt parts of the book for children, and modify the order of some topics, in order to make the book clearer.

The learning will be base on constructions and proofs. The constructions are useful for letting the child experience geometric ideas and get visual results. The proofs are a good way to understand geometry and are a good basis for future study of logic. Since the book is for children, we omit some of the proof details and use intuitive instead of precise definition. On the other hand, we insist on correct and elegant proofs. Precise definitions and exact proofs can be found in regular geometry books and can be used to extend to material to some of the children.

## Notation

The notation that is used in the book is defined at the first time it is used. However, it order to simplify the

use, it is also summarized in a special chapter.

## **Euclid's Elements online** **(<http://aleph0.clarku.edu/~djoyce/java/elements/toc.html>)**

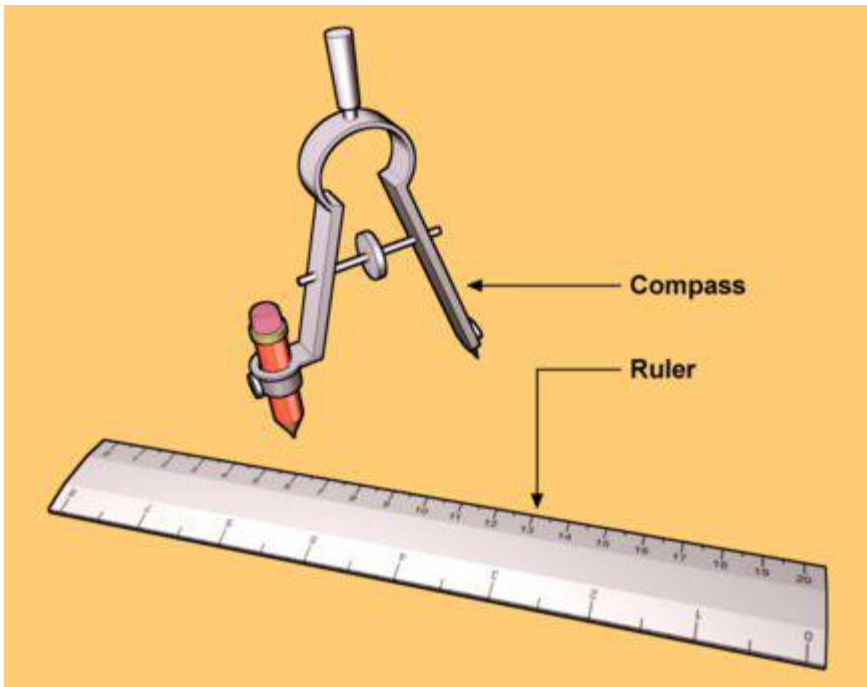
There is an wonderful online version of Euclid's Elements at this web site (<http://aleph0.clarku.edu/~djoyce/java/elements/toc.html>) . The site was created by David E. Joyce, a Professor of Mathematics and Computer Science at Clark University. This site includes all the text of the Elements, applets that display the constructions and many insightful comments. We give reference in this book to the original source and engource the reader to read the source in order to learn more about the chapter.

# Ruler and Compass

Geometry for elementary school		
<b>Introduction</b>	<b>Our tools: Ruler and compass</b>	<b>Points</b>

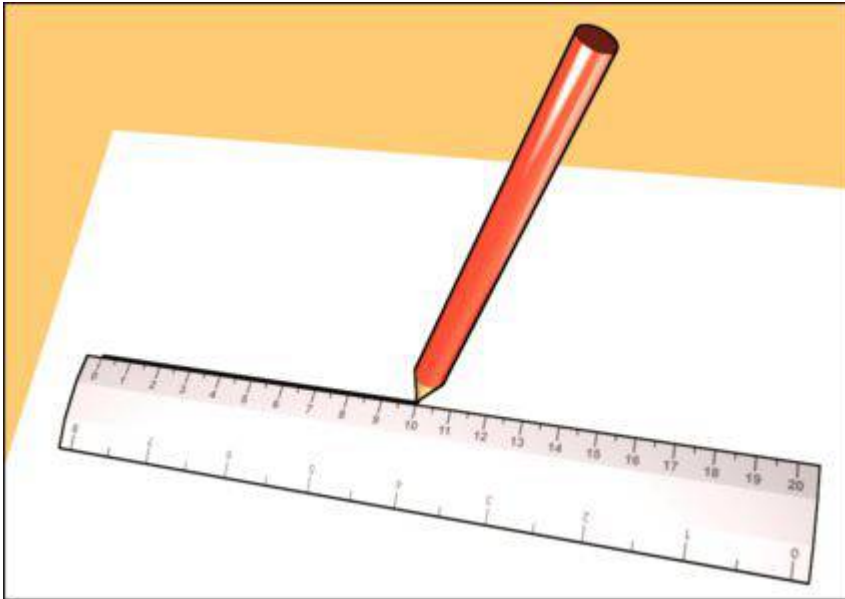
## Introduction

- Introduction of the tools



## How to draw a line?

- Using the ruler (based on Book I, Postulate 1 (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/post1.html>) ).



- We will use the notation  $\overline{AB}$  for the line segment that starts at **A** and ends at **B**. Note that we don't care about the segment direction and therefore  $\overline{AB}$  is the same as  $\overline{BA}$ .

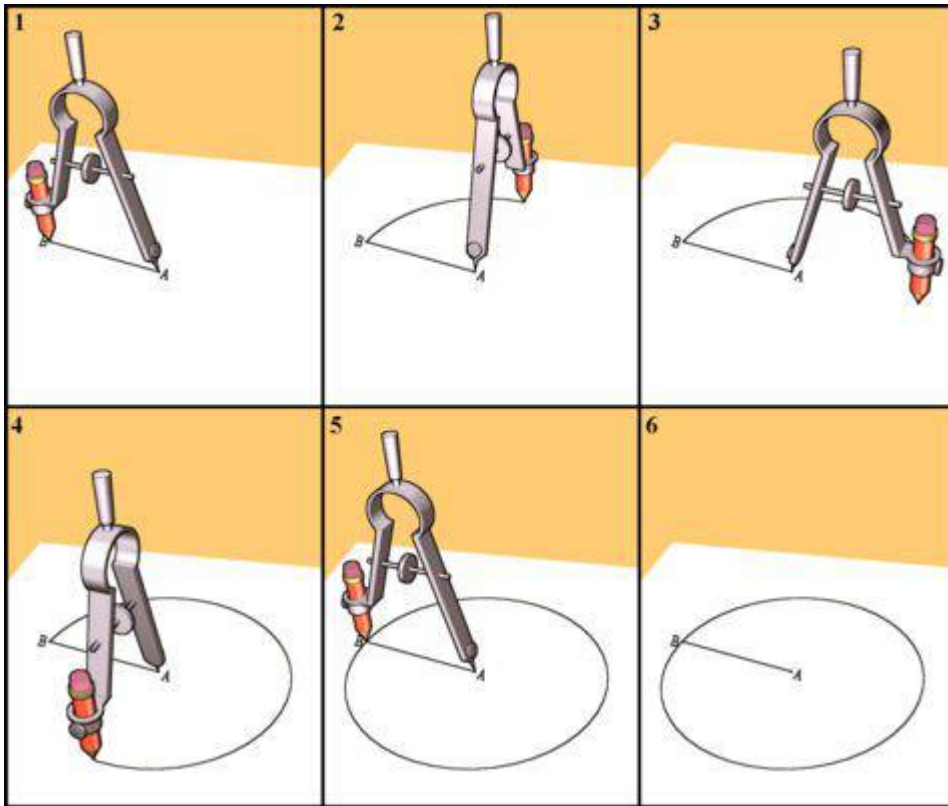
A ————— B

B ————— A

## How to draw a circle?

- Using the compass (based on Book I, Postulate 3 (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/post3.html>)).

We use the notation  $\circ A, \overline{AB}$  for the circle whose center is the point **A** and its radius length equals that of the segment  $\overline{AB}$ .



Note that in other sources, such as Euclid's Elements (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/defI15.html>) , a circle is described by any 3 points on its circumference, **ABC**.

The center-radius notation was chosen because of its suitability for constructing circles.



# Points

Geometry for elementary school		
<b>Our tools: Ruler and compass</b>	<b>Points</b>	<b>Lines</b>

- A **Point** is the limit of a circle whose size is decreasing.

You might be surprised to know, but this is not a point:



This is not a point.

The reason that this shape is not a point is that it is too large, it has area. This is a 'ball'.



This is not a point either.

Even when taking a ball of half that size we don't get a point.



Another non point.

And that is too large as well...

A point is so small that even if we divide the size of these balls by 100, 1,000 or 1,000,000 it will still be much larger than a point. A point is considered as *infinitely small*. In order to get to the size of a point we should keep dividing the ball size by two - forever. Don't try it at home - it will take you too much time!

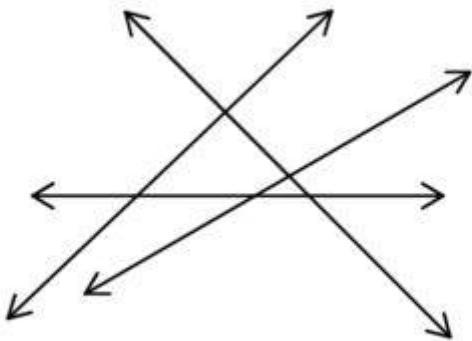
A point seems to be too small to be useful. Luckily, as we will see when discussing lines we have plenty of them.

# Lines

Geometry for elementary school		
<b>Points</b>	<b>Lines</b>	<b>Constructing equilateral triangle</b>

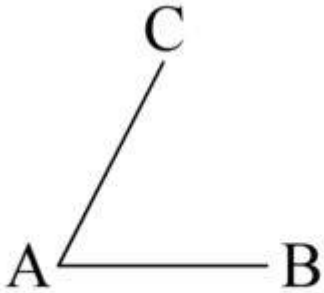
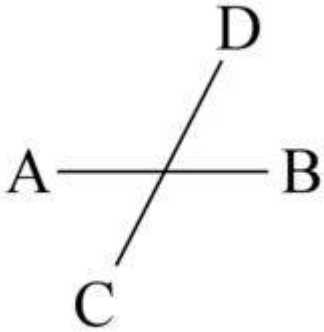
- non straight lines

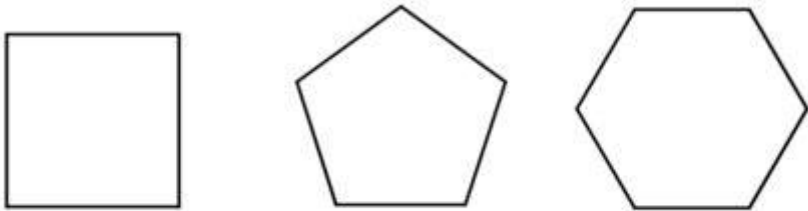
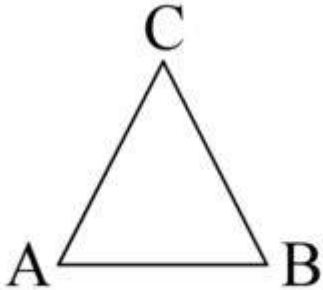
## Lines



A line is infinitely thin having infinite number of points, extending forever in both the directions. Two lines can intersect only in a single point.

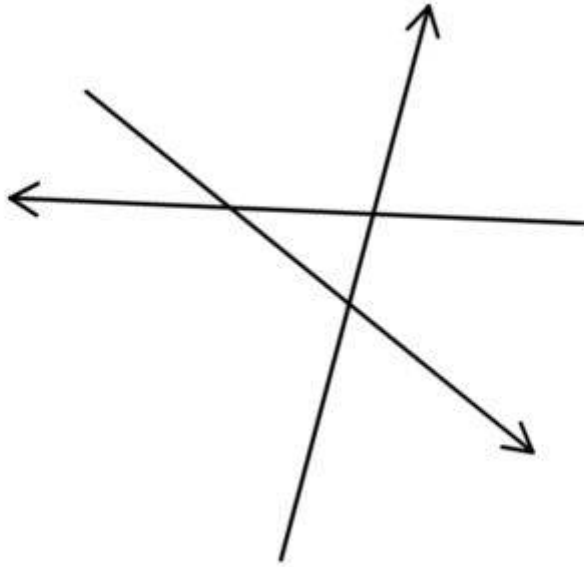
## Line segments





A line segment is a part of a line, which has two end points.

**Rays**



A ray has only one end point.

- axiom: there is only a single straight line between two points.

A.

·B



- show that this axiom is not true in general - draw straight lines on a ball

Points A and B on the surface of a sphere (a ball)



One way of connecting A and B with a straight line



Another way of connecting A and B with a straight line!



- show by halving that there are infinite number of points in a line
- show that the number of points in a long line and a short line is equal

# Constructing an Equilateral Triangle

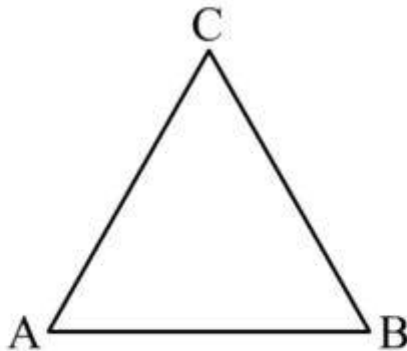
Geometry for elementary school		
Lines	Constructing equilateral triangle	Copying a line segment

## Introduction

In this chapter, we will show you how to draw an equilateral triangle. What does "equilateral" mean? It simply means that all three sides of the triangle are the same length.

Any triangle whose vertices (points) are **A**, **B** and **C** is written like this:  $\triangle ABC$ .

And if it's equilateral, it will look like the one in the picture below:



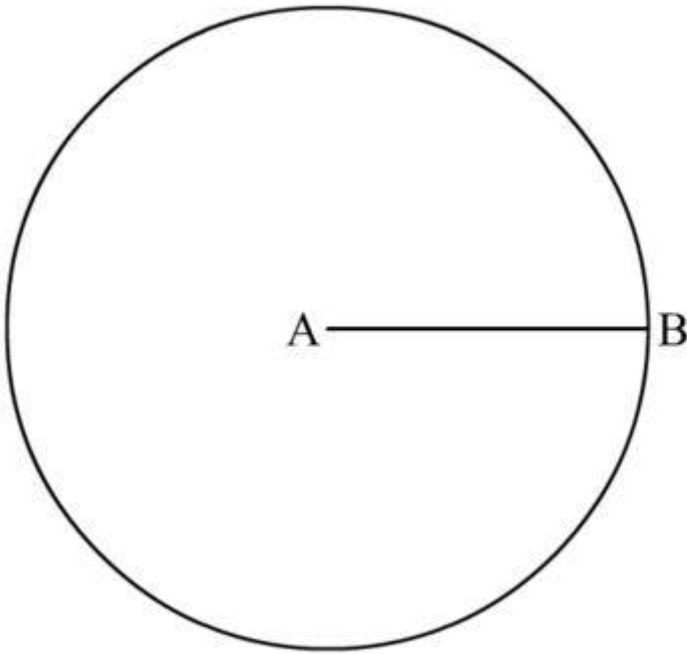
## The construction

The construction (method we use to draw it) is based on Book I, proposition 1 (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI1.html>) .

1. Using your ruler, Draw a line whatever length you want the sides of your triangle to be. Call one end of the line **A** and the other end **B**. Now you have a line segment called  $\overline{AB}$ . It should look something like the drawing below.

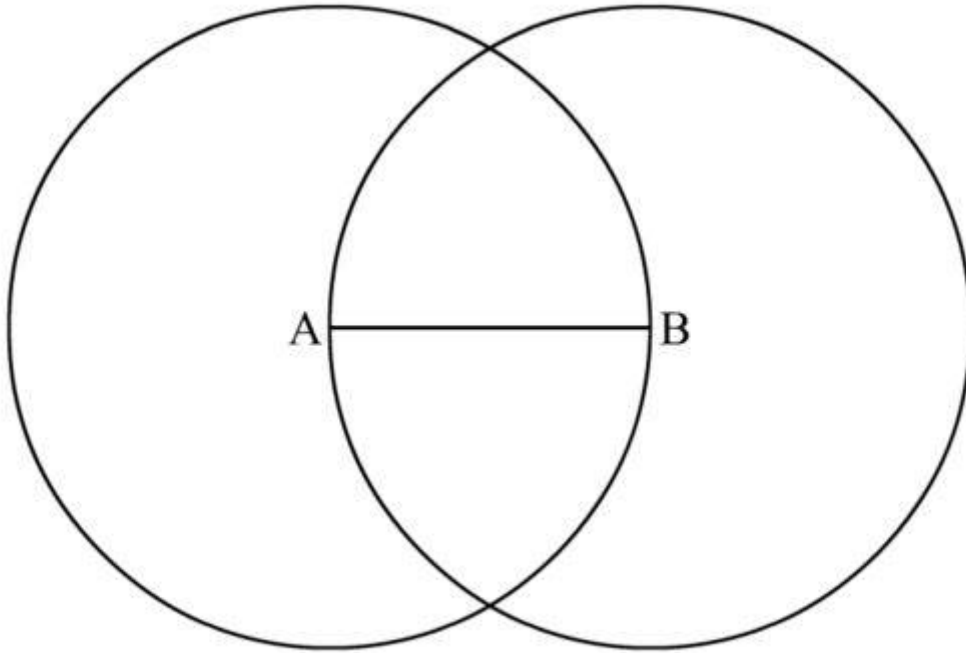


2. Using your compass, Draw the circle  $\circ A, \overline{AB}$ , whose center is **A** and radius is  $\overline{AB}$ .

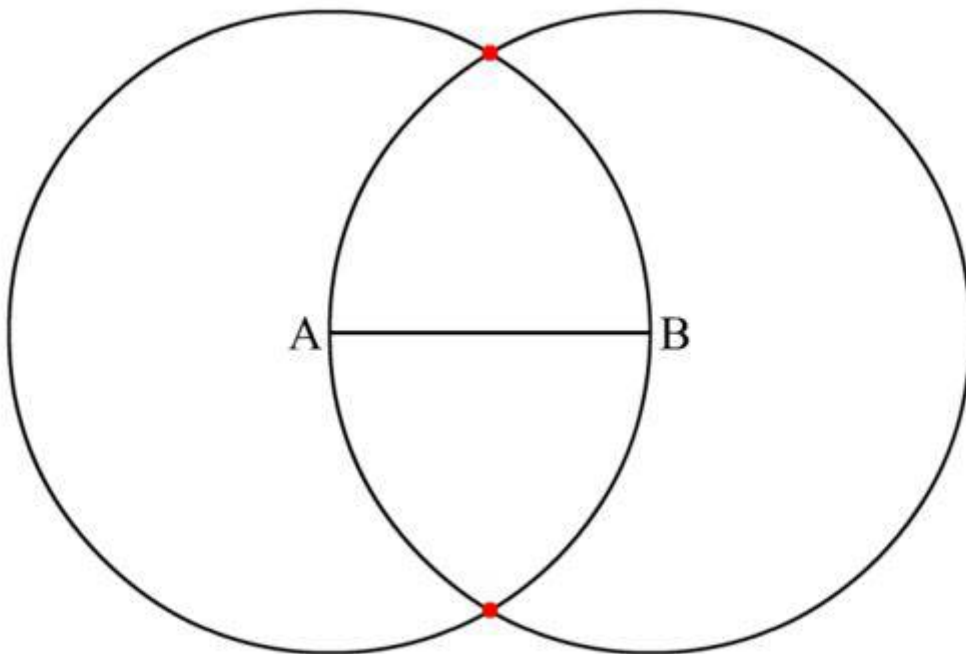


3. Again using your compass Draw the circle  $\circ B, \overline{AB}$ , whose center is **B** and radius is  $\overline{AB}$ .



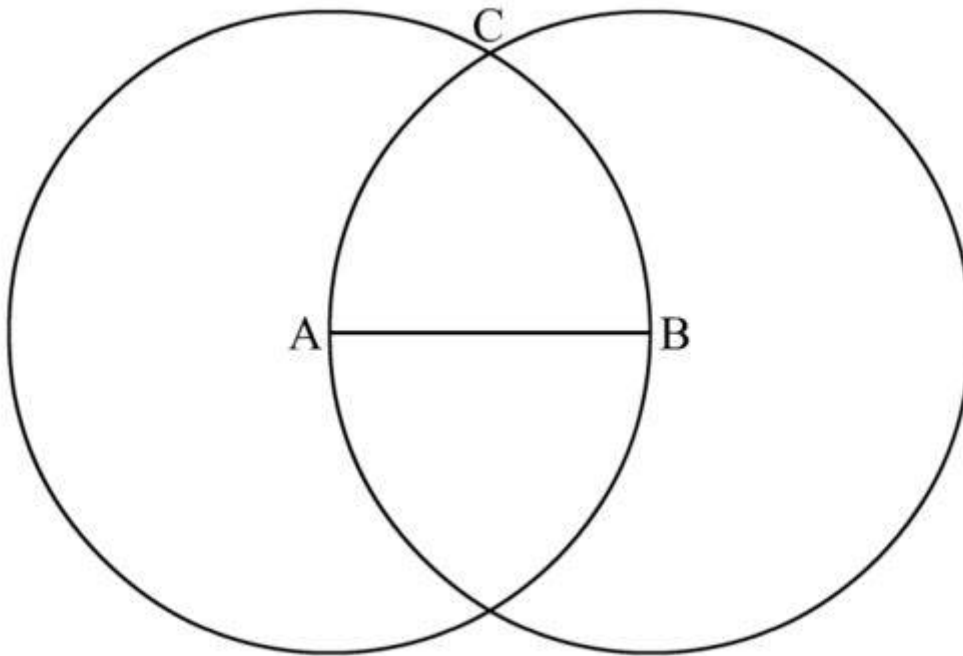


4. Can you see how the circles intersect (cross over each other) at two points?  
The points are shown in red on the picture below.

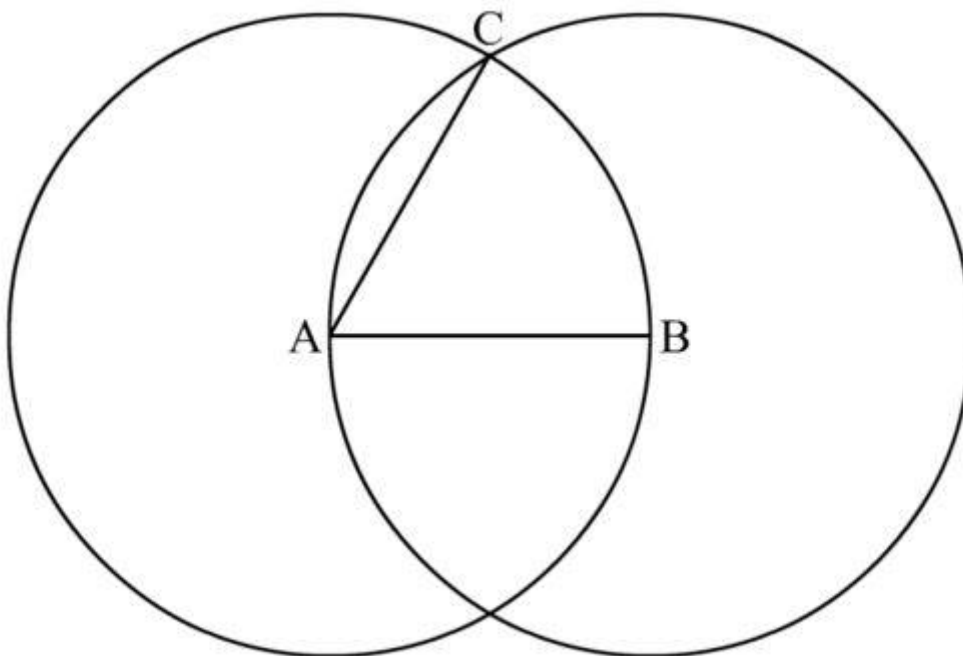


5. Choose one of these points and call it **C**.

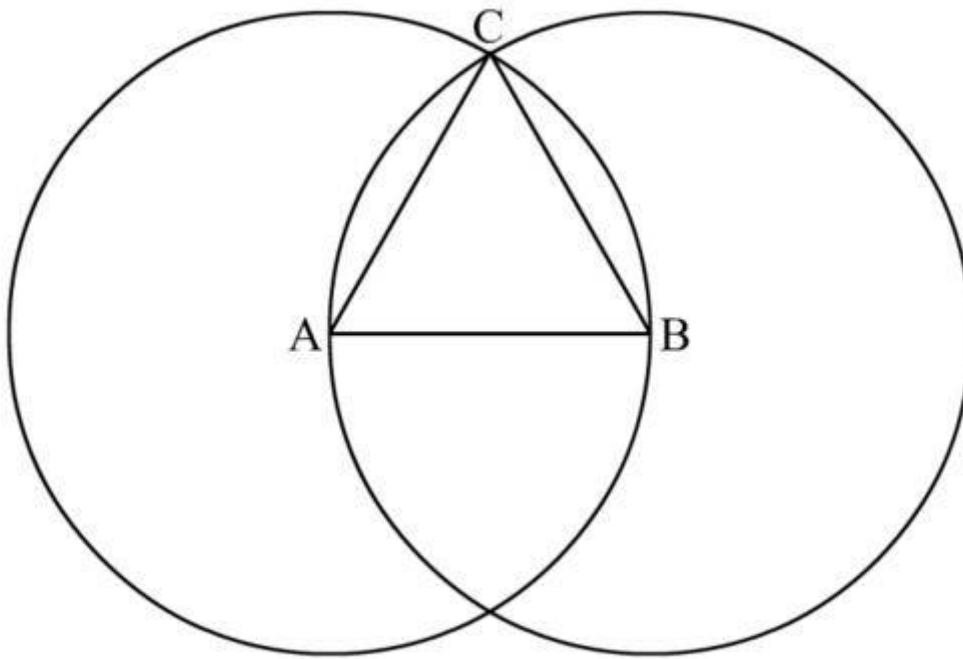
We chose the upper point, but you can choose the lower point if you like. If you choose the lower point, your triangle will look "upside-down", but it will still be an equilateral triangle.



6. Draw a line between **A** and **C** and get segment  $\overline{AC}$ .



7. Draw a line between **B** and **C** and get segment  $\overline{BC}$ .



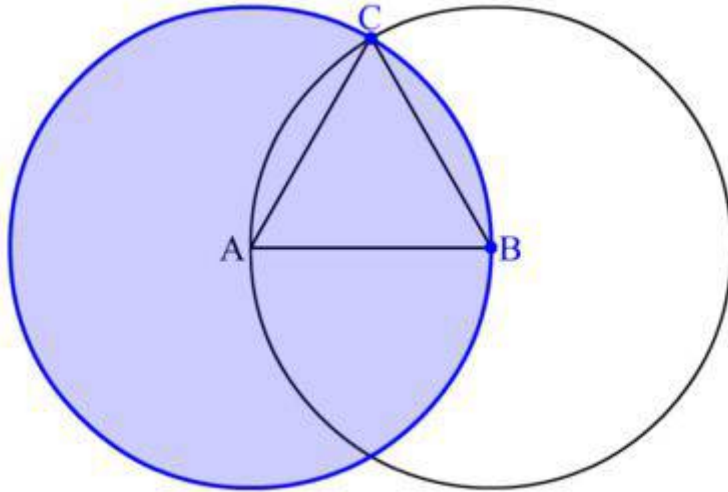
8. Construction completed.

## Claim

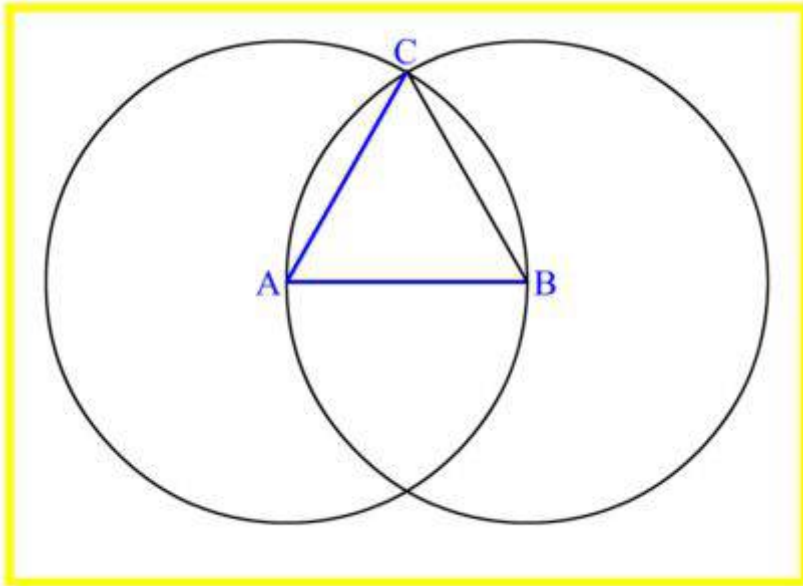
The triangle  $\triangle ABC$  is an equilateral triangle.

## Proof

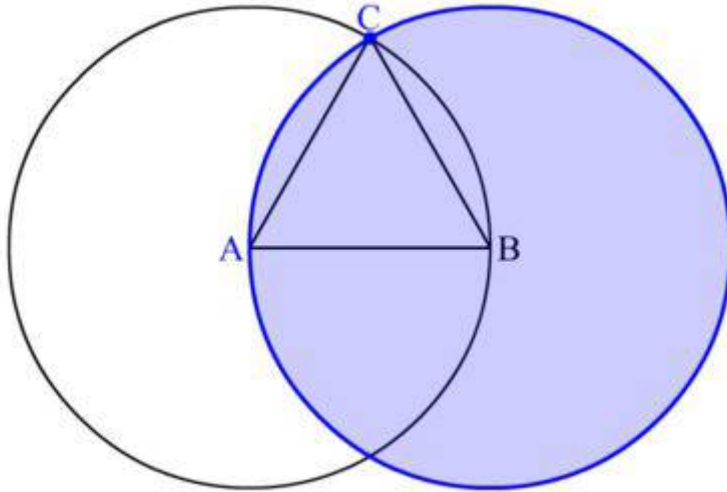
1. The points **B** and **C** are both on the circumference of the circle  $\circ A, \overline{AB}$  and point **A** is at the center.



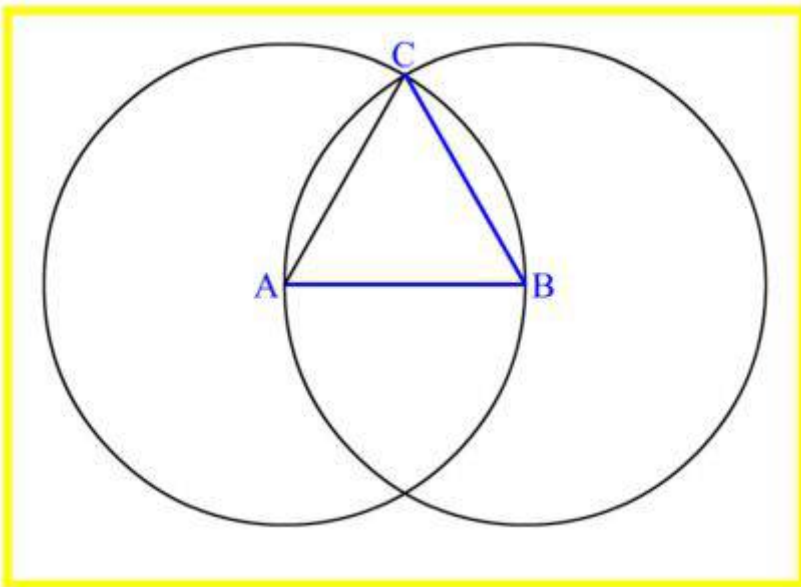
2. So the line  $\overline{AB}$  is the same length as the line  $\overline{AC}$ .  
Or, more simply,  $\overline{AB} = \overline{AC}$ .



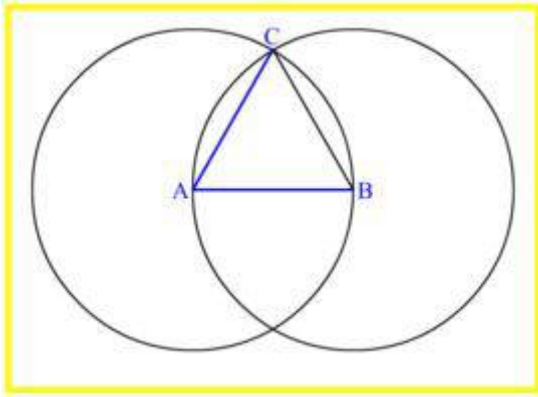
3. We do the same for the other circle:  
The points **A** and **C** are both on the circumference of the circle  $\circ B$ ,  $\overline{AB}$  and point **B** is at the center.



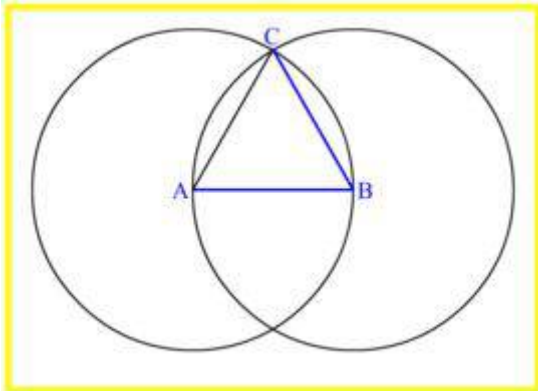
4. So we can say that  $\overline{AB} = \overline{BC}$ .



5. We've already shown that  $\overline{AB} = \overline{AC}$

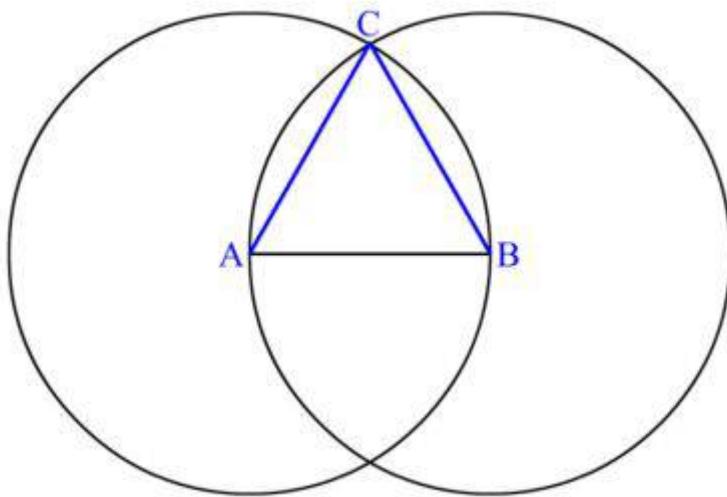


and  $\overline{AB} = \overline{BC}$ .



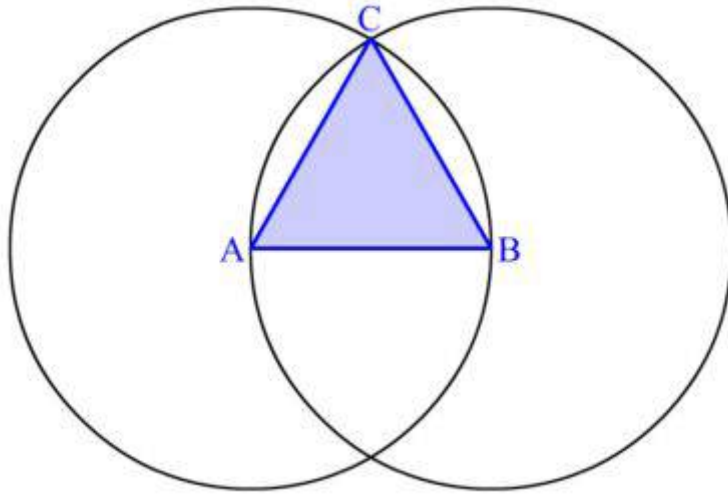
Since  $\overline{AC}$  and  $\overline{BC}$  are both equal in length to  $\overline{AB}$ , they must also be equal in length to each other.

So we can say  $\overline{AC} = \overline{BC}$



6. Therefore, the lines  $\overline{AB}$  and  $\overline{AC}$  and  $\overline{BC}$  are all equal.

$$\overline{AB} = \overline{AC} = \overline{BC}$$



7. We proved that all sides of  $\triangle ABC$  are equal, so this triangle is equilateral.

## Problems with the proof

The construction above is simple and elegant. One can imagine how children, using their legs as compass, accidentally find it.

However, Euclid's proof was wrong.

In mathematical logic, we assume some postulates. We construct proofs by advancing step by step. A proof should be made only of postulates and claims that can be deduced from the postulates. Some useful claims are given name and called theorems in order to enable to use them in future proofs.

There are some steps in its proof that cannot be deduced from the postulates. For example, according the postulates he used the circles  $\circ A, \overline{AB}$  and  $\circ B, \overline{AB}$  doesn't have to intersect.

Though that the proof was wrong, the construction is not necessarily wrong. One can make the construction valid, by extending the set of postulates. Indeed, in the years to come, different sets of postulates were proposed in order to make the proof valid. Using these sets, the construction that works so well on the using a pencil and papers, is also sound logically.

This error of Euclid, the gifted mathematician, should serves as an excellent example to the difficulty in mathematical proof and the difference between it and our intuition.

# Copying a Line Segment

Geometry for elementary school		
Constructing equilateral triangle	Copying a line segment	Copying a triangle

## Introduction

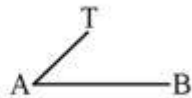
This construction copies a line segment  $\overline{AB}$  to a target point  $T$ . The construction is based on Book I, prop 2 (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI2.html>) .

## The construction

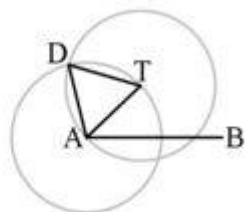
1. Let  $A$  be one of the end points of  $\overline{AB}$ . Note that we are just giving it a name here. (We could replace  $A$  with the other end point  $B$ ).



2. Draw a line  $\overline{AT}$

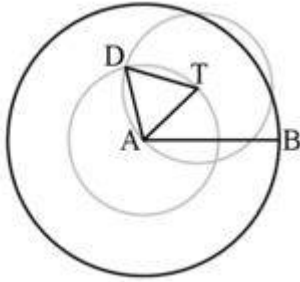


3. Construct an equilateral triangle  $\triangle ATD$  (a triangle that  $\overline{AT}$  is one of its sides).

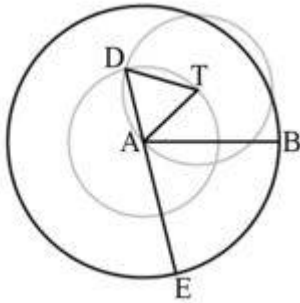




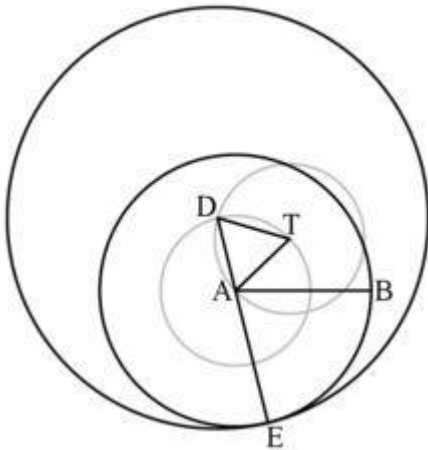
4. Draw the circle  $\circ A, \overline{AB}$ , whose center is **A** and radius is  $\overline{AB}$ .



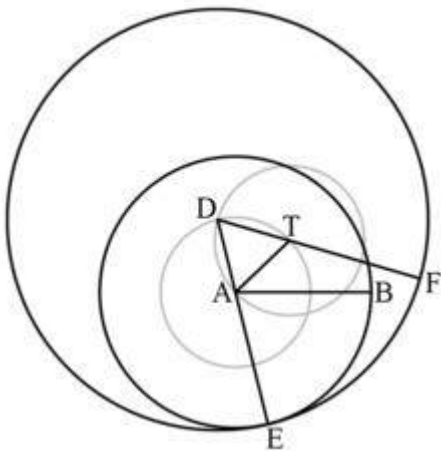
5. Draw a line starting from **D** going through **A** until it intersects  $\circ A, \overline{AB}$  and let the intersection point be **E**. Get segments  $\overline{AE}$  and  $\overline{DE}$ .



6. Draw the circle  $\circ D, \overline{DE}$ , whose center is **D** and radius is  $\overline{DE}$ .

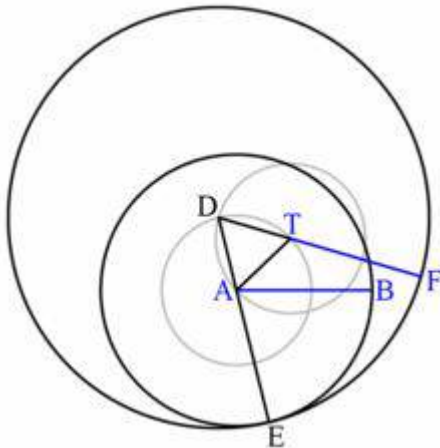


7. Draw a line starting from **D** going through **T** until it intersects  $\circ D, \overline{DE}$  and let the intersection point be **F**. Get segments  $\overline{TF}$  and  $\overline{DF}$ .



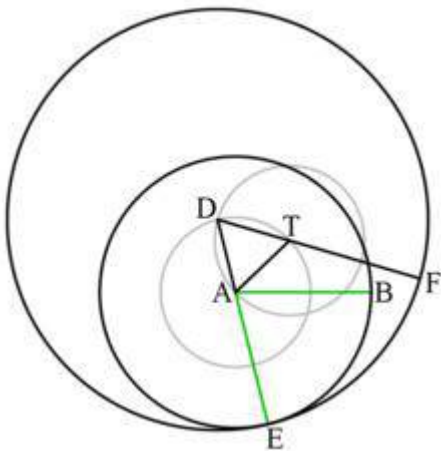
## Claim

The segment  $\overline{TF}$  is equal to  $\overline{AB}$  and starts at T.

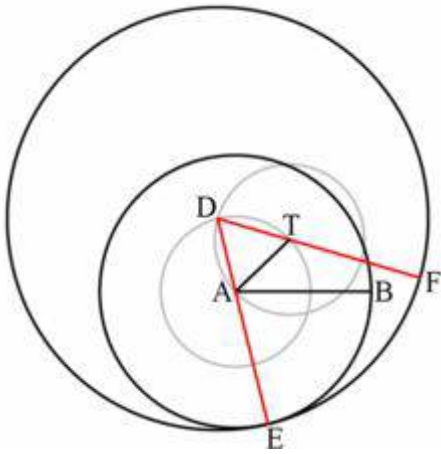


## Proof

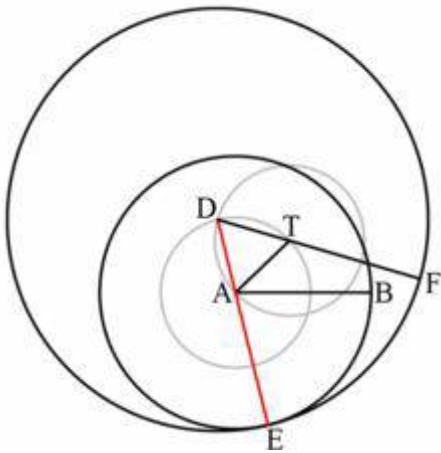
1. Segments  $\overline{AB}$  and  $\overline{AE}$  are both from the center of  $\circ A$ ,  $\overline{AB}$  to its circumference. Therefore they equal to the circle radius and to each other.



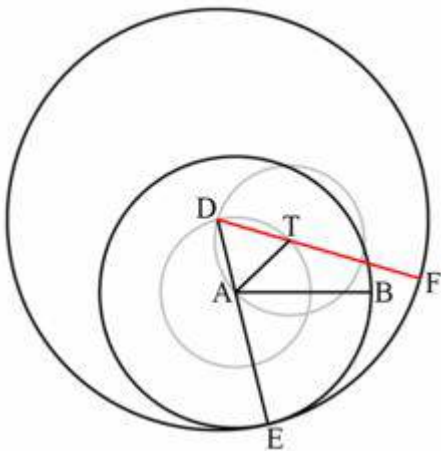
2. Segments  $\overline{DE}$  and  $\overline{DF}$  are both from the center of  $\circ D$ ,  $\overline{DE}$  to its circumference. Therefore they equal to the circle radius and to each other.



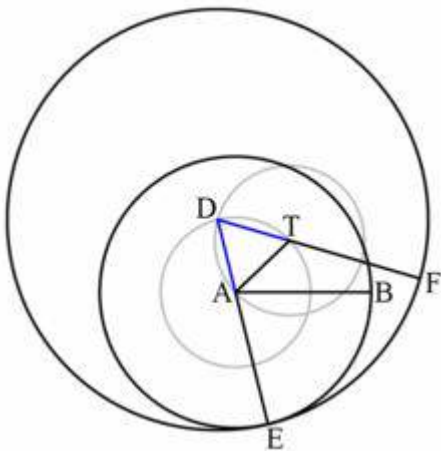
3.  $\overline{DE}$  equals to the sum of its parts  $\overline{DA}$  and  $\overline{AE}$ .



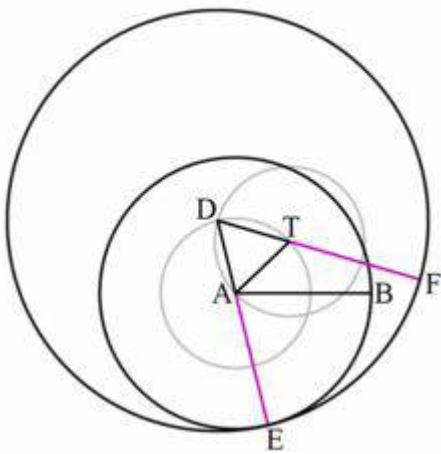
4.  $\overline{DF}$  equals to the sum of its parts  $\overline{DT}$  and  $\overline{TF}$ .



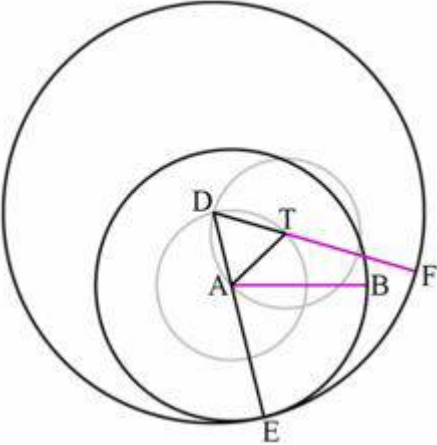
5. The segment  $\overline{DA}$  is equal to  $\overline{DT}$  since they are the sides of the equilateral triangle  $\triangle ATD$ .



6. Since the sum of segments is equal and two of the summands are equal so are the two other summands  $\overline{AE}$  and  $\overline{TF}$ .



7. Therefore  $\overline{AB}$  equals  $\overline{TF}$ .



# Constructing a Triangle

Geometry for elementary school		
Copying a line segment	Copying a triangle	Copying an angle

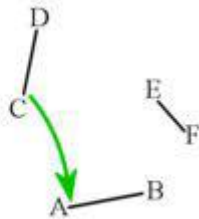
## Introduction

In this chapter, we will show how to construct a triangle from three segments. The construction is based on Book I, proposition 22 (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI22.html>)

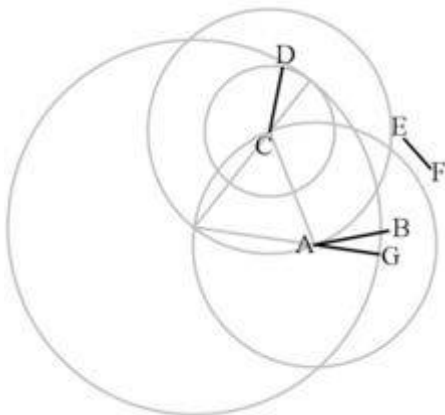
## The construction

Given three line segments  $\overline{AB}$ ,  $\overline{CD}$  and  $\overline{EF}$  we build a triangle whose sides equal the segments.

1. Copy the line  $\overline{CD}$  to point A.

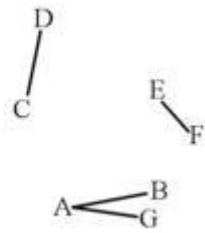


If you have forgotten how to do this, follow the instructions in the previous section. Your construction should look like the grey lines in the picture below. Call the new line  $\overline{AG}$

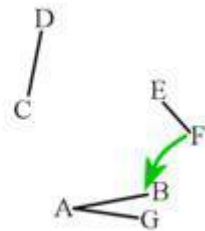


It's a good idea to erase your construction lines now, so all that's left are the four line segments

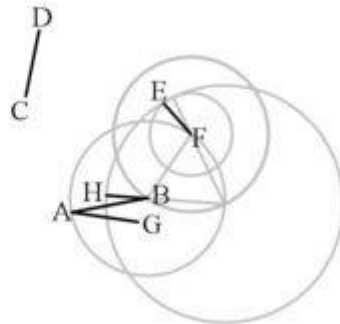
shown below.



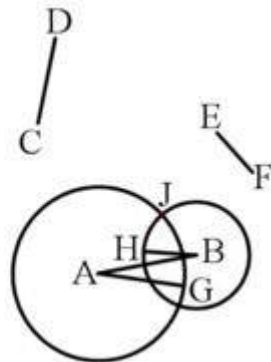
- Copy the line  $\overline{EF}$  to point **B**



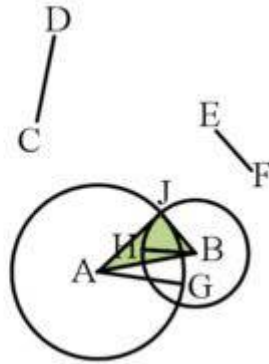
Your construction should look like the grey lines in the picture below. Call the new line  $\overline{BH}$



- Draw the circle  $\odot A, \overline{AG}$ , whose center is **A** and radius is  $\overline{AG}$ .
- Draw the circle  $\odot B, \overline{BH}$ , whose center is **B** and radius is  $\overline{BH}$ .
- Let **J** be an intersection point of  $\odot A, \overline{AG}$  and  $\odot B, \overline{BH}$ .



6. Draw a line  $\overline{AJ}$ .
7. Draw a line  $\overline{BJ}$ .



## Claim

The sides of the triangle  $\triangle ABJ$  equal to  $\overline{AB}$ ,  $\overline{CD}$  and  $\overline{EF}$ .

## Proof

1. The segment  $\overline{AB}$  is a side of the triangle and equal to itself.
2. The segment  $\overline{AJ}$  is equal to  $\overline{AG}$  because they are both radii of circle  $\odot A$ ,  $\overline{AG}$ . And because it was copied,  $\overline{AG} = \overline{CD}$ . Therefore  $\overline{AJ}$  is also equal to  $\overline{CD}$ .
3. The segment  $\overline{BJ}$  is equal to  $\overline{BH}$  because they are both radii of circle  $\odot B$ ,  $\overline{BH}$ . And because it was copied,  $\overline{BH} = \overline{EF}$ . Therefore  $\overline{BJ}$  is also equal to  $\overline{EF}$ .
4. Hence the sides of the triangle  $\triangle ABJ$  are equal to  $\overline{AB}$ ,  $\overline{CD}$  and  $\overline{EF}$ .

## Testing the procedure

1. Draw a line  $\overline{AB}$  of some length.
2. Copy the line  $\overline{AB}$  to an arbitrary point C and get  $\overline{AB}$ .
3. Draw a line  $\overline{EF}$  such that its length is three times the length of  $\overline{AB}$ . (We didn't specify how to construct such a segment and we give it as an exercise. Use chapter Copy the line as a guide for the solution.)
4. Construct a triangle from  $\overline{AB}$ ,  $\overline{CD}$  and  $\overline{EF}$ .

## Why you couldn't construct the triangle in the test?

The reason we couldn't build the triangle in the test is that the circles we constructed did not intersect.



One cannot use any three segments to construct a triangle. The length of the segments must obey a condition called "The triangle inequality". The triangle inequality states that any of the segments should be shorter than the sum of the length of the other two segments. If one of the segments is longer than the sum of the other two, the circles do not intersect. If one segment equals the sum of the other two, we get a line instead of a triangle.

Therefore, the construction is correct but one should condition the segments on which it can be applied. Note that the original construction was conditioned by Euclid, hence there is no error in the construction or in its proof.

# Why the Constructions are not Correct

Geometry for elementary school		
Copying an angle	Why the constructions are not correct?	The Side-Angle-Side congruence theorem

## Introduction

In the previous chapters, we introduced constructions and proved their correctness. Therefore, these constructions should work flawlessly. In this chapter, we will check whether the construction are indeed flawless.

## Testing a construction

1. Draw a line of  $\overline{AB}$  of length 10cm.
2. Copy the line segment to a different point **T**.
3. Measure the length of the segment you constructed.

## Explanation

I must admit that I never could copy the segment accurately. Some times the segment I constructed was of the length 10.5cm, I did even worse. A more talented person might get better results, but probably not exact.

How come the construction didn't work, at least in my case?

Our proof of the construction is correct. However, the construction is done in an ideal world. In this world, the lines and circles drawn are also ideal. They match the mathematical definition perfectly.

The circle I draw doesn't match the mathematical definition. Actually, many say that they don't match any definition of circle. When I try to use the construction, I'm using the wrong building blocks.

However, the construction are not useless in our far from ideal world. If we use approximation of a circle in the construction, we are getting and approximation of the segment copy. After all, Even my copy is not too far from the original.

## Note

In the Euclidian geometry developed by the Greek the rule is used only to draw lines. One cannot measure the length of segments using the rulers as we did in this test. Therefore our test should be viewed as a

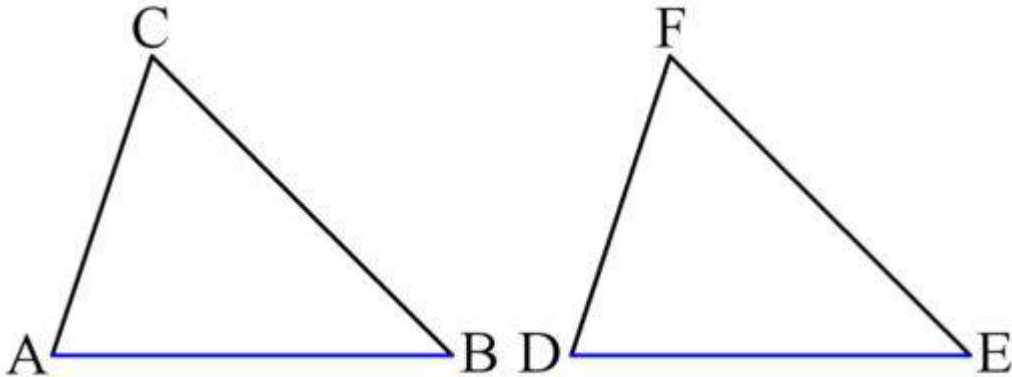
criticism of the use of Euclidian geometry in the real world and not as part of that geometry.

# Side-Side-Side Congruence Theorem

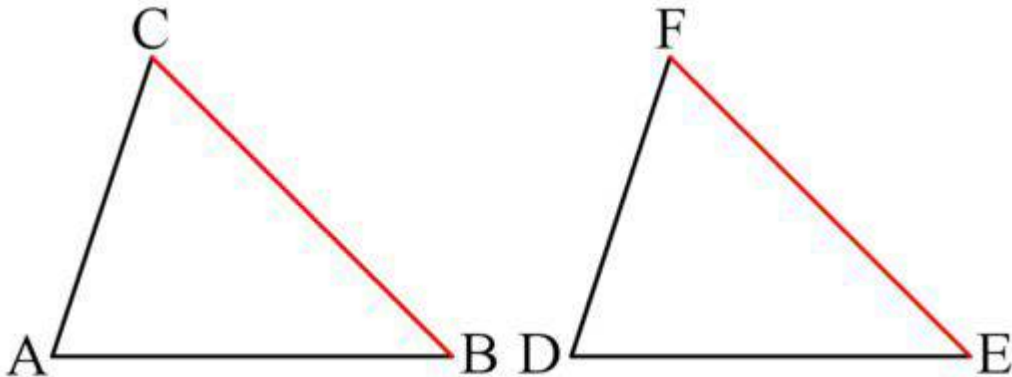
## Introduction

In this chapter, we will start the discussion of congruence and congruence theorems. We say the two triangles are congruent if they have the same shape. The triangles  $\triangle ABC$  and  $\triangle DEF$  congruence if and only if all the following conditions hold:

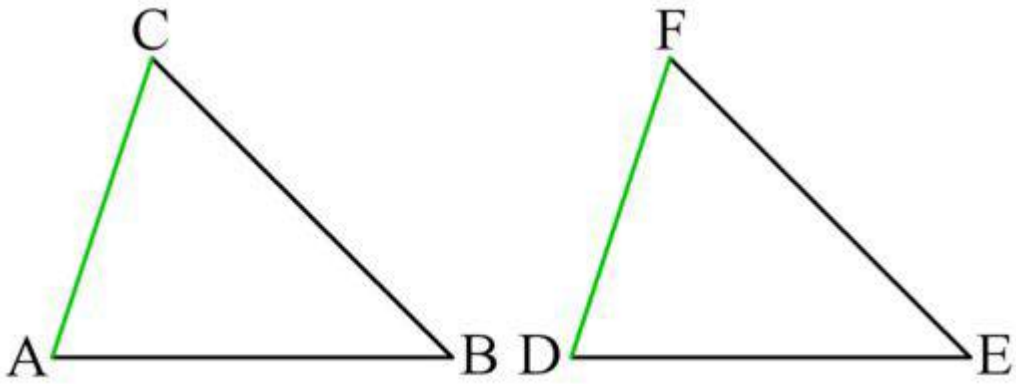
1. The side  $\overline{AB}$  equals  $\overline{DE}$ .



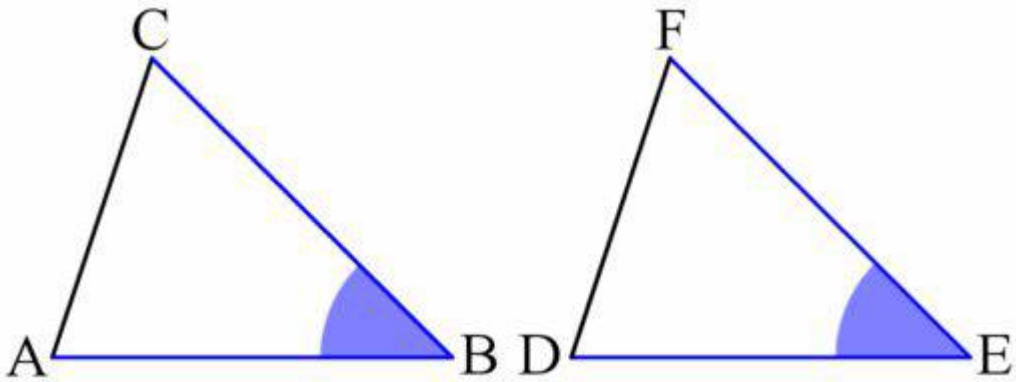
2. The side  $\overline{BC}$  equals  $\overline{EF}$ .



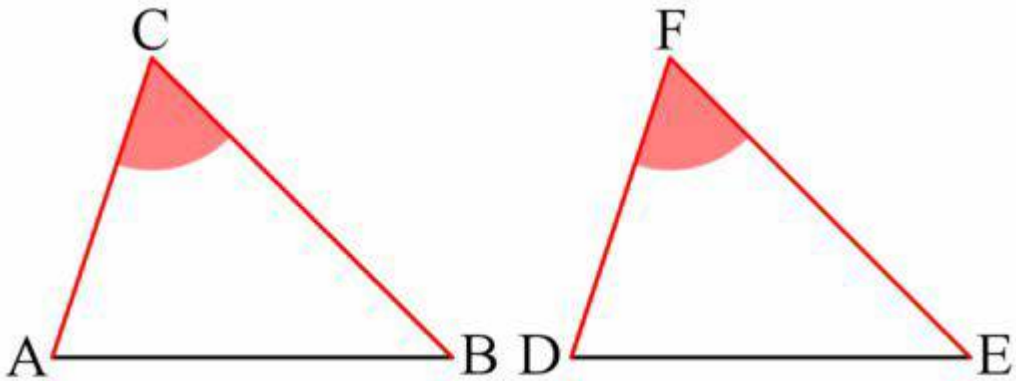
3. The side  $\overline{AC}$  equals  $\overline{DF}$ .



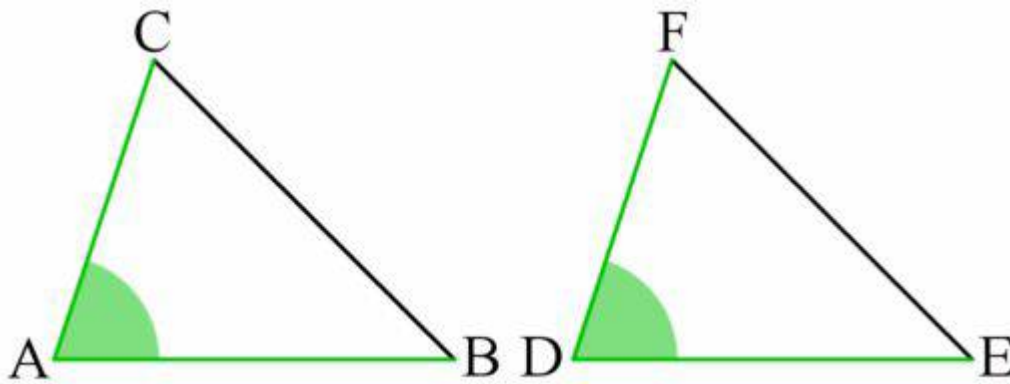
4. The angle  $\angle ABC$  equals  $\angle DEF$ .



5. The angle  $\angle BCA$  equals  $\angle FED$ .



6. The angle  $\angle CAB$  equals  $\angle FDE$ .



Note that the order of vertices is important. It is possible that  $\triangle ABC$  and  $\triangle ACB$  won't congruence though it is the same triangle.

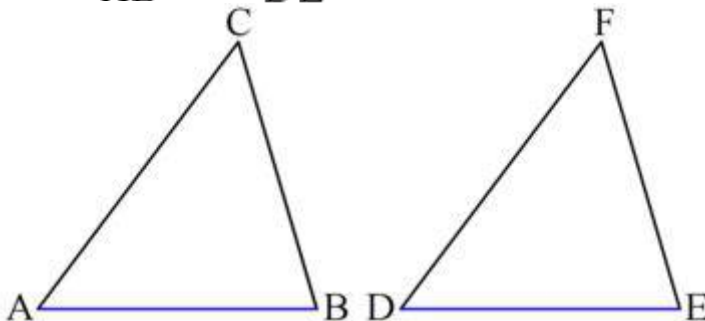
Congruence theorems give a set of less conditions that are sufficient in order to show that two triangles congruence.

The first congruence theorem we will discuss is the Side-Side-Side theorem.

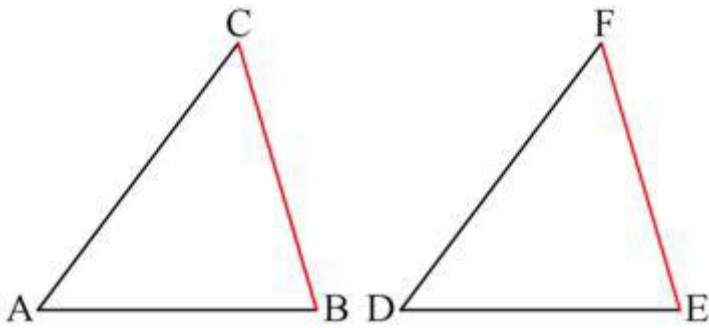
## The Side-Side-Side congruence theorem

Given two triangles  $\triangle ABC$  and  $\triangle DEF$  such that their sides are equal, hence:

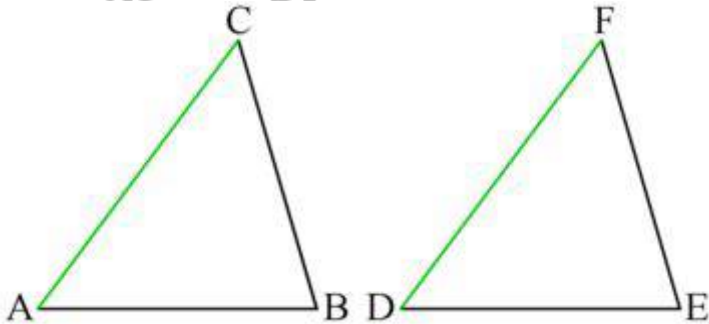
1. The side  $\overline{AB}$  equals  $\overline{DE}$ .



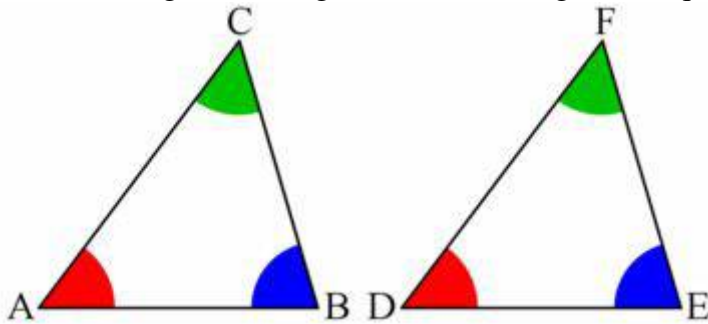
2. The side  $\overline{BC}$  equals  $\overline{EF}$ .



3. The side  $\overline{AC}$  equals  $\overline{DF}$ .



Then the triangles are congruent and their angles are equal too.



## Method of Proof

In order to prove the theorem we need a new postulate. The postulate is that one can move or flip any shape in the plane without changing it. In particular, one can move a triangle without changing its sides or angles. Note that this postulate is true in plane geometry but not in general. If one considers geometry over a ball, the postulate is no longer true.

Given the postulate, we will show how can we move one triangle to the other triangle location and show

that they coincide. Due to that, the triangles are equal.

### The construction

1. Copy The line Segment side  $\overline{AB}$  to the point **D**.
2. Draw the circle  $\circ D, \overline{AB}$ .
3. The circle  $\circ D, \overline{AB}$  and the segment  $\overline{DE}$  intersect at the point **E** hence we have a copy of  $\overline{AB}$  such that it coincides with  $\overline{DE}$ .
4. Construct a triangle with  $\overline{DE}$  as its base,  $\overline{BC}$ ,  $\overline{AC}$  as the sides and the vertex at the side of the vertex **F**. Call this triangle triangles  $\triangle DEG$

### The claim

The triangles  $\triangle DEF$  and  $\triangle ABC$  congruent.

### The proof

1. The points **A** and **D** coincide.
2. The points **B** and **E** coincide.
3. The vertex **F** is an intersection point of  $\circ D, \overline{DF}$  and  $\circ E, \overline{EF}$ .
4. The vertex **G** is an intersection point of  $\circ D, \overline{AC}$  and  $\circ E, \overline{BC}$ .
5. It is given that  $\overline{DF}$  equals  $\overline{AC}$ .
6. It is given that  $\overline{EF}$  equals  $\overline{BC}$ .
7. Therefore,  $\circ D, \overline{DF}$  equals  $\circ D, \overline{AC}$  and  $\circ E, \overline{EF}$  equals  $\circ E, \overline{BC}$ .
8. However, circles of different centers has at most one intersection point in one side of the segment the joins their centers.
9. Hence, the points **G** and **F** coincide.
10. Thorough two points only an straight line passes, Therefore  $\overline{EG}$  coincides with  $\overline{EF}$  and  $\overline{GD}$  coincides with  $\overline{DF}$ .
11. Therefore, the  $\triangle DEG$  coincides with  $\triangle DEF$  and therefore congruent.
12. Due to the postulate  $\triangle DEG$  and  $\triangle ABC$  are equal and therefore congruent.
13. Hence,  $\triangle DEF$  and  $\triangle ABC$  congruent.
14. Hence,  $\angle ABC$  equals  $\angle DEF$ ,  $\angle BCA$  equals  $\angle EFD$  and  $\angle CAB$  equals  $\angle FDE$ .

### Note

The Side-Side-Side congruence theorem appears as Book I, prop 8 (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI8.html>) at the Elements. The proof here is in



the spirit of the original proof. In the original proof Euclid claims that the vertices **F** and **G** must coincide but doesn't show why. We used the assumption that "circles of different centers has at most one intersection point in one side of the segment the joins their centers". This assumption is true in plane geometry but doesn't follows from Euclid's original postulates. Since Euclid himself had to use such assumption, we preferred to give a more detaild proof, though the extra assumption.

# Copying a Triangle

Geometry for elementary school		
Copying a line segment	Copying a triangle	Copying an angle

## Introduction

In this chapter, we will show how to copy a triangle  $\triangle ABC$  to other triangle  $\triangle CDE$ . The construction is an excellent example of the **reduction** technique – solving a problem by solution to a previously solved problem.

## The construction

1. Construct a triangle from the sides of  $\triangle ABC$ :  $\overline{AB}$ ,  $\overline{AC}$ ,  $\overline{BC}$  and get  $\triangle CDE$ .

## Claim

The triangles  $\triangle ABC$  and  $\triangle CDE$  congruence.

## Proof

1.  $\overline{AB}$ ,  $\overline{AC}$ ,  $\overline{BC}$  are sides of the triangle  $\triangle ABC$  and therefore obey the triangle inequality.
2. Therefore one can build a triangle whose sides equal these segments.
3. The sides of the triangle  $\triangle ABC$  and  $\triangle CDE$  are equal.
4. Due to the Side-Side-Side congruence theorem the triangles  $\triangle ABC$  and  $\triangle CDE$  congruence.

# Copying an Angle

Geometry for elementary school		
Copying a triangle	Copying an angle	Why the constructions are not correct?

## Introduction

In this chapter, we will show how to copy an angle  $\angle ABC$  to other angle  $\angle CDE$ . The construction is based on I, proposition 23 (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI23.html>Book).

## The construction

1. Draw a line between **A** and **B** and get  $\triangle ABC$ .
2. Copy the triangle  $\triangle ABC$  and get  $\triangle CDE$

## Claim

The angles  $\angle ABC$  and  $\angle CDE$  are equal.

## Proof

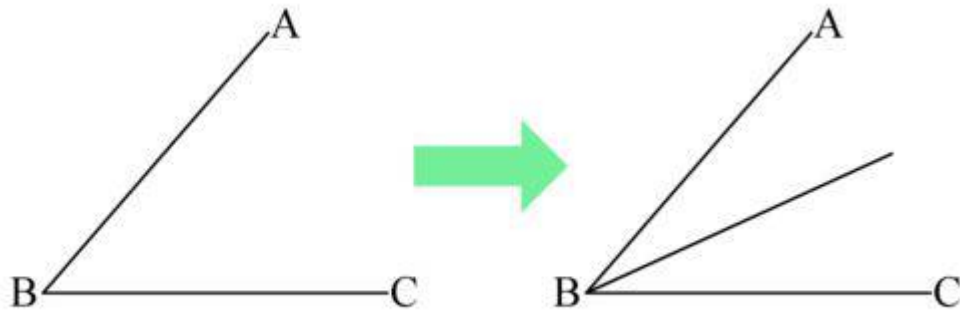
1. The triangles  $\triangle ABC$  and get  $\triangle CDE$  congruence.
2. Therefore the angles of the triangles equal.
3. Hence,  $\angle ABC$  and  $\angle CDE$  are equal.

Note that any two points on the rays can be used to create a triangle.

# Bisecting an Angle

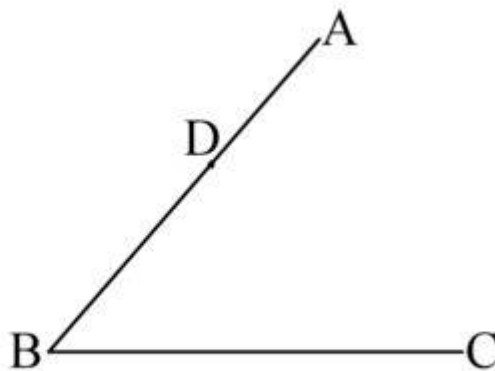
## Introduction

In this chapter, we will learn how to bisect an angle. Given an angle  $\angle ABC$  we will divide it to two equal angles. The construction is based on book I, proposition 9 (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI9.html>)

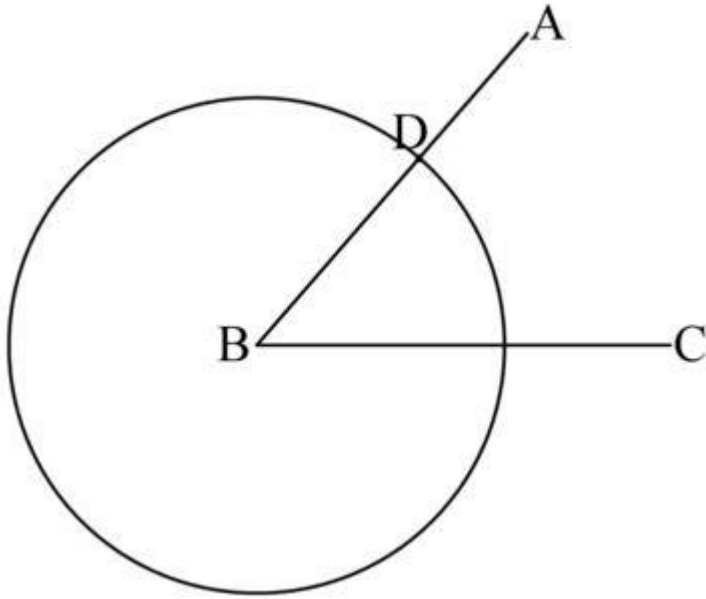


## The construction

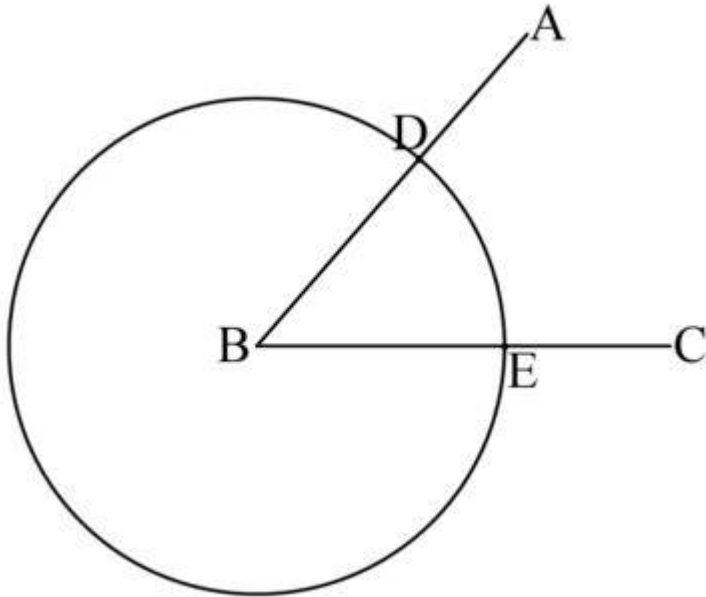
1. Choose an arbitrary point  $D$  on the segment  $\overline{AB}$ .



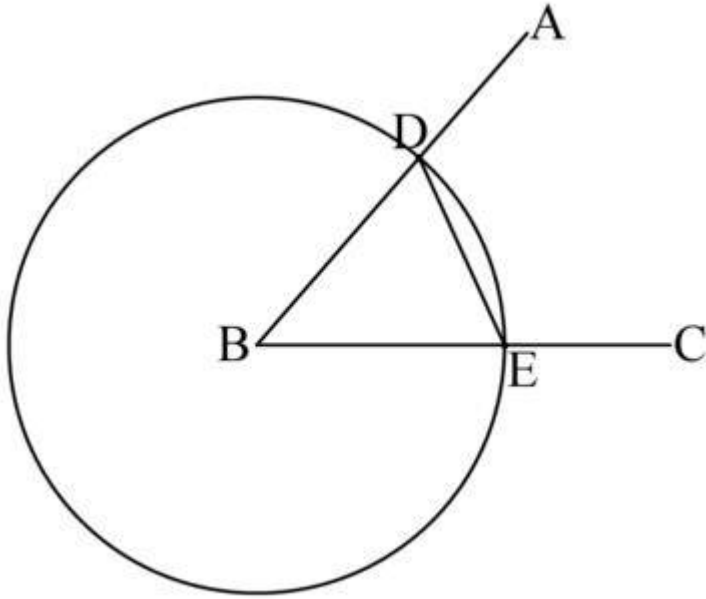
2. Draw the circle  $\circ B, \overline{BD}$ .



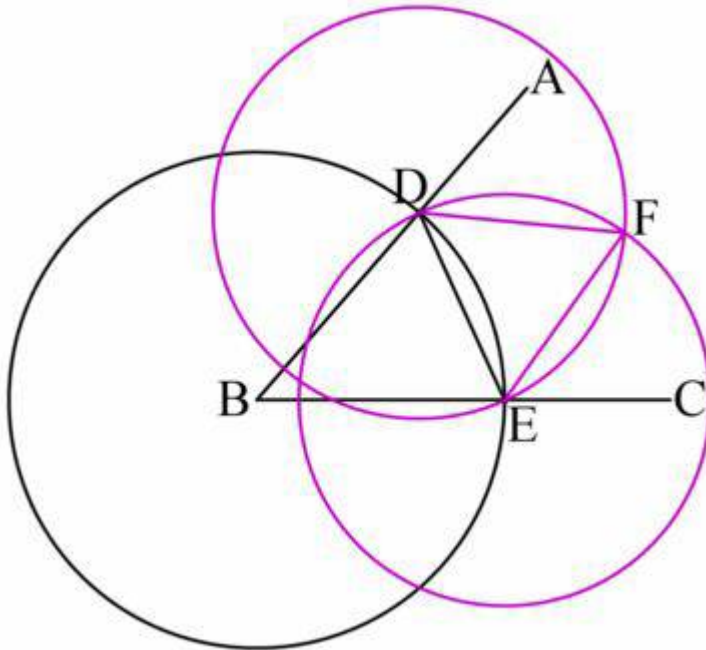
3. Let **E** be the intersection point of  $\overline{BC}$  and  $\odot B, \overline{BD}$ .



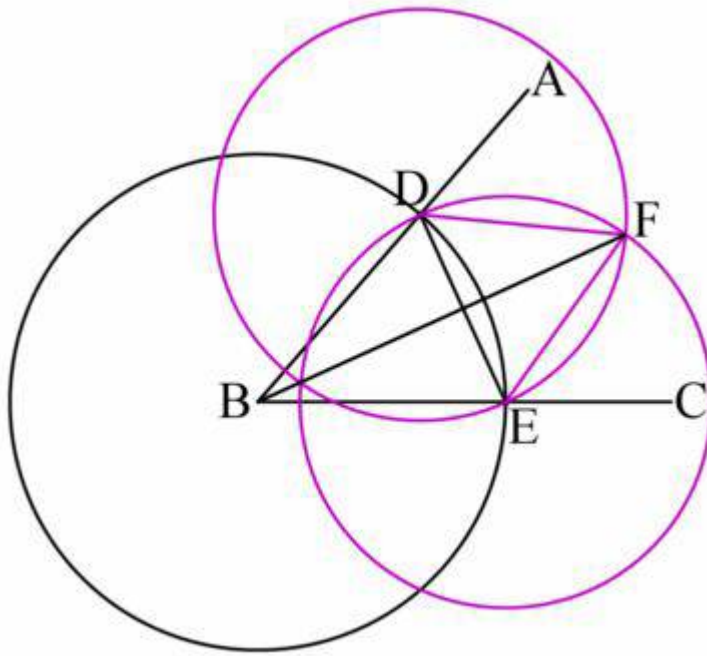
4. Draw the line  $\overline{DE}$ .



5. Construct an equilateral triangle on  $\overline{DE}$  with third vertex  $F$  and get  $\triangle DEF$ .



6. Draw the line  $\overline{BF}$ .



## Claim

1. The angles  $\angle ABF$  ,  $\angle FBC$  equal to half of  $\angle ABC$ .

## The proof

1.  $\overline{DE}$  is a segment from the center to the circumference of  $\circ B$ ,  $\overline{BD}$  and therefore equals its radius.
2. Hence,  $\overline{BE}$  equals  $\overline{BD}$  .
3.  $\overline{DF}$  and  $\overline{EF}$  are sides of the equilateral triangle  $\triangle DEF$ .
4. Hence,  $\overline{DF}$  equals  $\overline{EF}$  .
5. The segment  $\overline{BF}$  equals to itself
6. Due to the Side-Side-Side congruence theorem the triangles  $\triangle ABF$  and  $\triangle FBC$  congruent.
7. Hence, the angles  $\angle ABF$  ,  $\angle FBC$  equal to half of  $\angle ABC$ .

## Note

We showed a simple method to divide an angle to two. A natural question that rises is how to divide an angle into other numbers. Since Euclid's days, mathematicians looked for a method for trisecting an angle, dividing it into 3. Only after years of trials it was proven that no such method exists since such a

construction is impossible, using only ruler and compass.

## Exercise

1. Find a construction for dividing an angle to 4.
2. Find a construction for dividing an angle to 8.
3. For which other number you can find such constructions?



# Side-Angle-Side Congruence Theorem

Geometry for elementary school		
Why the constructions are not correct?	The Side-Angle-Side congruence theorem	Some impossible constructions

## Introduction

In this chapter, we will discuss another congruence theorem, this time the Side-Angle-Side theorem. The theorem appears as Based on Book I, prop 4 (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI4.html>) at the Elements.

## The Side-Angle-Side congruence theorem

Given two triangles  $\triangle ABC$  and  $\triangle DEF$  such that their sides are equal, hence:

1. The side  $\overline{AB}$  equals  $\overline{DE}$ .
2. The side  $\overline{CA}$  equals  $\overline{DF}$ .
3. The angle  $\angle CAB$  equals  $\angle FDE$  (These are the angles between the sides).

Then the triangles congruent and their other angles and side are equal too.

## Proof

We will use the method of superposition – we will move one triangle to the other one and we will show that they coincide. We won't use the construction we learned to copy a line or a segment but we will move the triangle as whole.

1. Superpose  $\triangle ABC$  on  $\triangle DEF$  such that **A** is place on **D** and  $\overline{AB}$  is placed on  $\overline{DE}$ .
2. It is given that  $\overline{AB}$  equals  $\overline{DE}$ .
3. Hence, **B** coincides with **E**.
4. It is given that the angle  $\angle CAB$  equals  $\angle FDE$ .
5. Hence,  $\overline{CA}$  is placed on  $\overline{DF}$ .
6. it is given that  $\overline{CA}$  equals  $\overline{DF}$ .
7. Hence, **C** coincides with **F**.
8. Therefore,  $\overline{CB}$  coincides with  $\overline{EF}$ .

9. The triangles  $\triangle ABC$  and  $\triangle DEF$  coincide.
10. The triangles  $\triangle ABC$  and  $\triangle DEF$  congruent.

# Bisecting a Segment

## Introduction

In this chapter, we will learn how to bisect a segment. Given a segment  $\overline{AB}$ , we will divide it to two equal segments  $\overline{AC}$  and  $\overline{CB}$ . The construction is based on book I, proposition 10 (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI10.html>).

## The construction

1. Construct the equilateral triangle  $\triangle ABD$  on  $\overline{AB}$ .
2. Bisect an angle on  $\angle ADB$  using the segment  $\overline{DE}$ .
3. Let  $C$  be the intersection point of  $\overline{DE}$  and  $\overline{AB}$ .

## Claim

1. Both  $\overline{AC}$  and  $\overline{CB}$  are equal to half of  $\overline{AB}$ .

## The proof

1.  $\overline{AD}$  and  $\overline{BD}$  are sides of the equilateral triangle  $\triangle ABD$ .
2. Hence,  $\overline{AD}$  equals  $\overline{BD}$ .
3. The segment  $\overline{DC}$  equals to itself.
4. Due to the construction  $\angle ADE$  and  $\angle EDB$  are equal.
5. The segments  $\overline{DE}$  and  $\overline{DC}$  lie on each other.
6. Hence,  $\angle ADE$  equals to  $\angle ADC$  and  $\angle EDB$  equals to  $\angle CDB$ .
7. Due to the Side-Angle-Side congruence theorem the triangles  $\triangle ADC$  and  $\triangle CDB$  congruent.
8. Hence,  $\overline{AC}$  and  $\overline{CB}$  are equal.
9. Since  $\overline{AB}$  is the sum of  $\overline{AC}$  and  $\overline{CB}$ , each of them equals to its half.

# Some Impossible Constructions

Geometry for elementary school		
<b>The Side-Angle-Side congruence theorem</b>	<b>Some impossible constructions</b>	<b>Parallel lines</b>

## Introduction

In the pervious chapters, we discussed several construction procedures. In this chapter, we will number some problems for which there is no construction using only ruler and compass.

The problems were introduced by the Greek and since then mathematicians tried to find constructions for them. Only in 1882, it was proven that there is no construction for the problems.

Note that the problems have no construction when we restrict our self to constructions using ruler and compass. The problems can be solved when allowing the use of other tools or operations, for example, if we use Origami (<http://www.merrimack.edu/~thull/omfiles/geoconst.html>) .

The mathematics involved in proving that the constructions are impossible are more too advanced for this book. Therefore, we only name the problems and give reference to the proof of their impossibility at the further reading section.

## Impossible constructions

### Squaring the circle

The problem is to find a construction procedure that in a finite number of steps, to make a square with the same area as a given circle.

### Doubling the cube

To "double the cube" means to be given a cube of some side length  $s$  and volume  $V$ , and to construct a new cube, larger than the first, with volume  $2V$  and therefore side length  $\sqrt[3]{2s}$ .

### Trisecting the angle

The problem is to find a construction procedure that in a finite number of steps, constructs an angle that is

one-third of a given arbitrary angle.

## Further reading

Proving that the constructions are impossible involve mathematics that is not in the scope of this book.

The interested reader can use these links to learn why the constructions are impossible.

The Four Problems Of Antiquity (<http://www.cut-the-knot.org/arithmetic/antiquity.shtml>) has no solution since their solution involves constructing a number that is not a constructible number (<http://www.cut-the-knot.org/arithmetic/rational.shtml>) . The numbers that should have being constructed in the problems are defined by these cubic Equations (<http://www.cut-the-knot.org/arithmetic/cubic.shtml>) .

It is recommended to read the references in this order:

1. Four Problems Of Antiquity (<http://www.cut-the-knot.org/arithmetic/antiquity.shtml>)
2. Constructible numbers (<http://www.cut-the-knot.org/arithmetic/rational.shtml>)
3. Cubic Equations (<http://www.cut-the-knot.org/arithmetic/cubic.shtml>)

# Pythagorean Theorem

Write about [Pythagorean theorem](http://en.wikipedia.org/wiki/Pythagorean_theorem) ([http://en.wikipedia.org/wiki/Pythagorean\\_theorem](http://en.wikipedia.org/wiki/Pythagorean_theorem)) and its use to prove that  $\sqrt{0.5}$  is an irrational number ([http://en.wikipedia.org/wiki/Irrational\\_number](http://en.wikipedia.org/wiki/Irrational_number)) .

## Introduction

In this chapter, we will discuss the Pythagorean theorem. It is used to find the side lengths of right triangles. It says:

*In any right triangle, the area of the square whose side is the hypotenuse (the side of a right triangle opposite the right angle) is equal to the sum of areas of the squares whose sides are the two legs (i.e. the two sides other than the hypotenuse).*

This means that if  $\triangle ABC$  is a right triangle, the length of the hypotenuse,  $c$ , squared equals the sum of  $a$  squared plus  $b$  squared. Or:

$$a^2 + b^2 = c^2$$

Here's an example:

In a right-angled triangle,  $a=5\text{cm}$  and  $b=7\text{cm}$ , so what is  $c$ ?

$$\begin{aligned} a^2 + b^2 &= c^2 \\ c &= \sqrt{5^2 + 7^2} \\ c &= \sqrt{25 + 49} \\ c &= \sqrt{74} \\ c &= 8.6 \end{aligned}$$

If  $c$  is **not larger** than  $a$  or  $b$ , your answer is incorrect.

# Parallel Lines

Geometry for elementary school		
<b>Some impossible constructions</b>	<b>Parallel lines</b>	<b>Squares</b>

## Definition

The definition of parallel lines is based on Book I, definition 23 (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/defI23.html>) .

Parallel lines are straight lines that never intersect. Notice that when we consider parallel segments we require that there is no intersection point even if we extend the line the segments lie on.

## The parallel lines postulate

The postulate appears in Euclid's elements as the fifth postulate (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/post5.html>) .

Let there be two lines. If there is a third line that intersects them such that the sum of the interior angles on one side is smaller than two right angles then the two lines intersect.

This postulate was suspect as redundant. Mathematicians though that instead of assuming it, the postulate can be deduced from other postulates. However, the attempts to deduce this postulate failed. The reason to this failure is that the indeed, the parallel postulate doesn't follow from the other ones. While we assume it in plane geometry, one can define different geometries (e.g., on a ball) for which this postulate is not valid.

# Squares

<b>Geometry for elementary school</b>		
<b>Parallel lines</b>	<b>Squares</b>	<b>Notation</b>

A square is a geometric figure comprised of four lines of equal length, which are connected at right angles.

- Prove properties of different squares using parallel lines theorems.
- Show that the more properties we have the more we can prove but on less shapes.



# A Proof of Irrationality

In mathematics, a rational number is a real number that can be written as the ratio of two integers, i.e., it is of the form

$a/b$  where  $a$  and  $b$  are integers and  $b$  is not zero. An **irrational number** is a number that cannot be written as a ratio of two integers, i.e., it is not of the form  $a/b$ .

## History of the theory of irrational numbers

The discovery of irrational numbers is usually attributed to Pythagoras, more specifically to the Pythagorean Hippasus of Metapontum, who produced a proof of the irrationality of the  $\sqrt{2}$ . The story goes that Hippasus discovered irrational numbers when trying to represent the square root of 2 as a fraction (proof below). However, Pythagoras believed in the absoluteness of numbers, and could not accept the existence of irrational numbers. He could not disprove their existence through logic, but his beliefs would not accept the existence of irrational numbers and so he sentenced Hippasus to death by drowning. As you see, mathematics might be dangerous.

## Irrationality of the square root of 2

One proof of the irrationality of the square root of 2 is the following proof by contradiction. The proposition is proved by assuming the negation and showing that that leads to a contradiction, which means that the proposition must be true.

The term coprime is used in the proof. Two integers are coprimes if none of them divides the other.

1. Assume that  $\sqrt{2}$  is a rational number. This would mean that there exist integers  $a$  and  $b$  such that  $a/b = \sqrt{2}$ .
2. Then  $\sqrt{2}$  can be written as an irreducible fraction (the fraction is shortened as much as possible)  $a/b$  such that  $a$  and  $b$  are coprime integers and  $(a/b)^2 = 2$ .
3. It follows that  $a^2/b^2 = 2$  and  $a^2 = 2b^2$ .
4. Therefore  $a^2$  is even because it is equal to  $2b^2$  which is obviously even.
5. It follows that  $a$  must be even. (Odd numbers have odd squares and even numbers have even squares.)
6. Because  $a$  is even, there exists a  $k$  that fulfills:  $a = 2k$ .
7. We insert the last equation of (3) in (6):  $2b^2 = (2k)^2$  is equivalent to  $2b^2 = 4k^2$  is equivalent to  $b^2 = 2k^2$ .

8. Because  $2k^2$  is even it follows that  $b^2$  is also even which means that  $b$  is even because only even numbers have even squares.
9. By (5) and (8)  $a$  and  $b$  are both even, which contradicts that  $a / b$  is irreducible as stated in (2).

Since we have found a contradiction the assumption (1) that  $\sqrt{2}$  is a rational number must be false. The opposite is proven.  $\sqrt{2}$  is irrational.

# Fractals

## Introduction

All the previous constructions we considered had one thing in common. The constructions were ended after a final number of steps. When one recalls that mathematicians actually used a ruler and compass in order to execute the constructions, this requirement seems to be in place. However, when we remove this requirement we can construct new interesting geometric shapes. In this chapter we will introduce two of them. Note that these shapes are not part of Euclidian geometry and were considered only years after its development.

## Cantor Set

For a full overview of Cantor set see the article ([http://en.wikipedia.org/wiki/Cantor\\_set](http://en.wikipedia.org/wiki/Cantor_set)) at wikipedia on which this section is based. The **Cantor set** was introduced by German mathematician Georg Cantor.

The Cantor set is defined by repeatedly removing the middle thirds of line segments. One starts by removing the middle third from the unit interval  $[0, 1]$ , leaving  $[0, 1/3] \cup [2/3, 1]$ . Next, the "middle thirds" of all of the remaining intervals are removed. This process is continued for ever. The Cantor set consists of all points in the interval  $[0, 1]$  that are not removed at any step in this infinite process.

### What's in the Cantor set?

Since the Cantor set is defined as the set of points not excluded, the proportion of the unit interval remaining can be found by total length removed. This total is the geometric series

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \dots = \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \left( \frac{1}{1 - \frac{2}{3}} \right) = 1.$$

So that the proportion left is  $1 - 1 = 0$ . Alternatively, it can be observed that each step leaves  $2/3$  of the length in the previous stage, so that the amount remaining is  $2/3 \times 2/3 \times 2/3 \times \dots$ , an infinite product which equals 0 in the limit.

From the calculation, it may seem surprising that there would be anything left — after all, the sum of the lengths of the removed intervals is equal to the length of the original interval. However a closer look at the process reveals that there must be something left, since removing the "middle third" of each interval involved removing open sets (sets that do not include their endpoints). So removing the line segment  $(1/3, 2/3)$  from the original interval  $[0, 1]$  leaves behind the points  $1/3$  and  $2/3$ . Subsequent steps do not remove these (or other) endpoints, since the intervals removed are always internal to the intervals remaining. So we know for certain that the Cantor set is not empty.

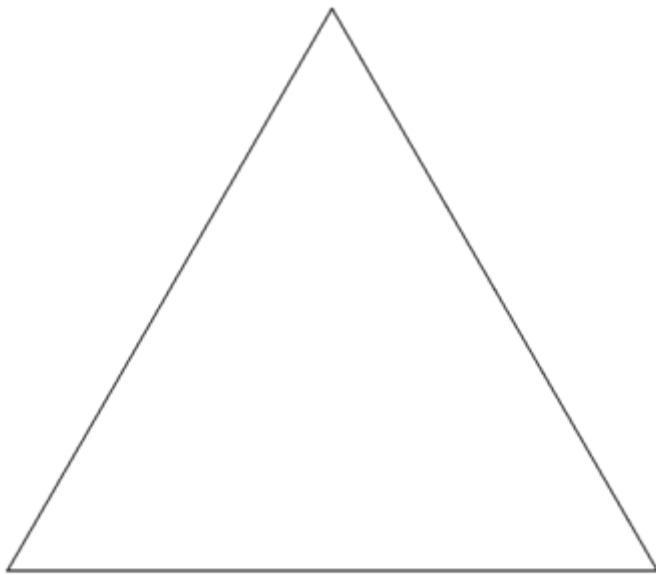
## The Cantor set is a fractal

The Cantor set is the prototype of a fractal (<http://en.wikipedia.org/wiki/Fractal>) . It is self-similar (<http://en.wikipedia.org/wiki/Self-similar>) , because it is equal to two copies of itself, if each copy is shrunk by a factor of  $1/3$  and translated.

## Koch curve

For a full overview of Koch curve see the article ([http://en.wikipedia.org/wiki/Koch\\_curve](http://en.wikipedia.org/wiki/Koch_curve)) at wikipedia on which this section is based.

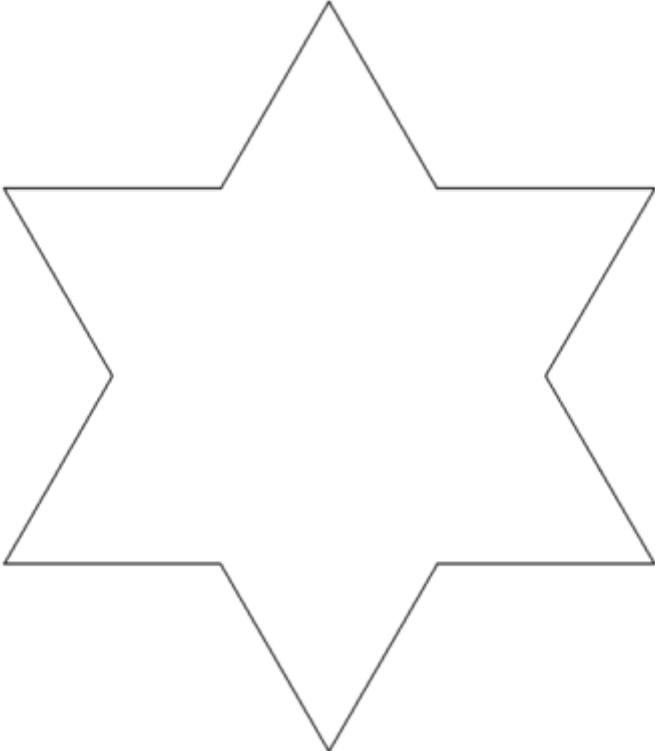
The **Koch curve** is a one of the earliest fractal (<http://en.wikipedia.org/wiki/Fractal>) curves to have been described. It was published during 1904 by the Swedish mathematician Helge von Koch. The better known **Koch snowflake** (or **Koch star**) is the same as the curve, except it starts with an equilateral triangle instead of a line segment.



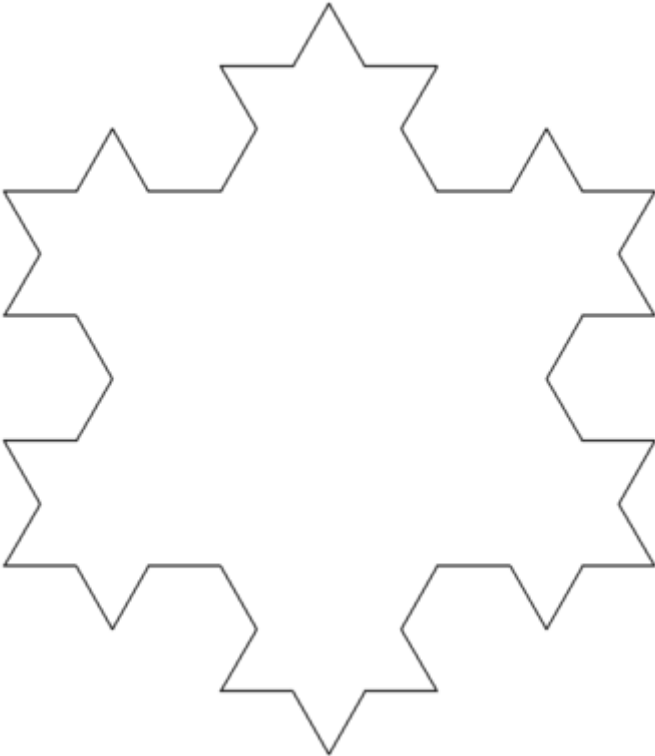
One can imagine that it was created by starting with a line segment, then recursively altering each line segment as follows:

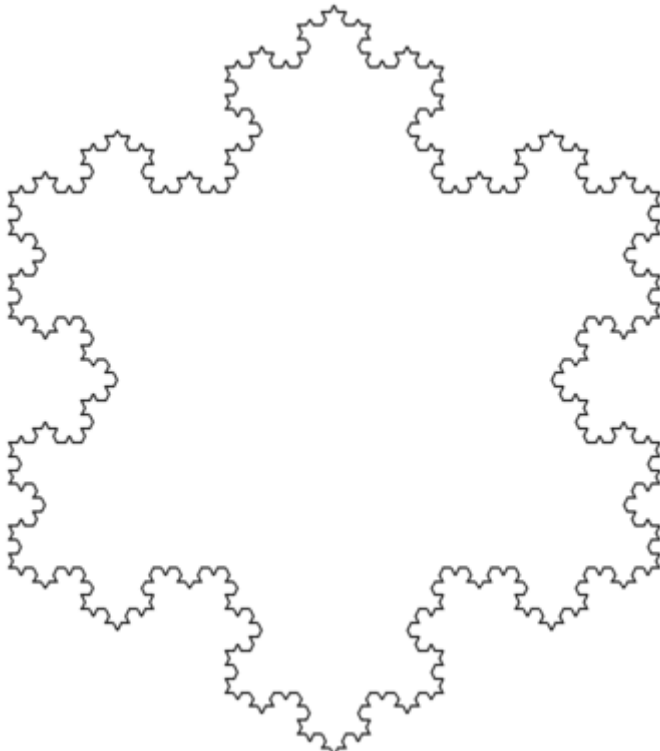
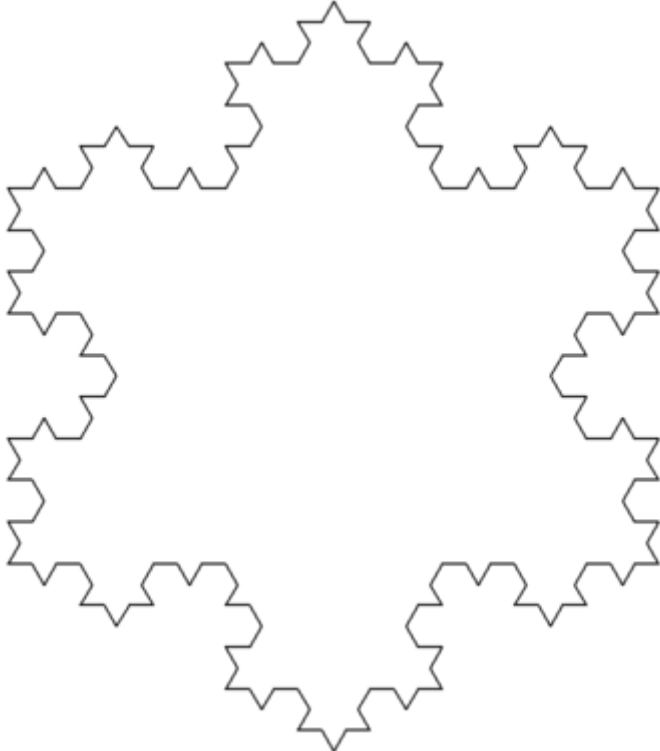
1. divide the line segment into three segments of equal length.
2. draw an equilateral triangle that has the middle segment from step one as its base.
3. remove the line segment that is the base of the triangle from step 2.

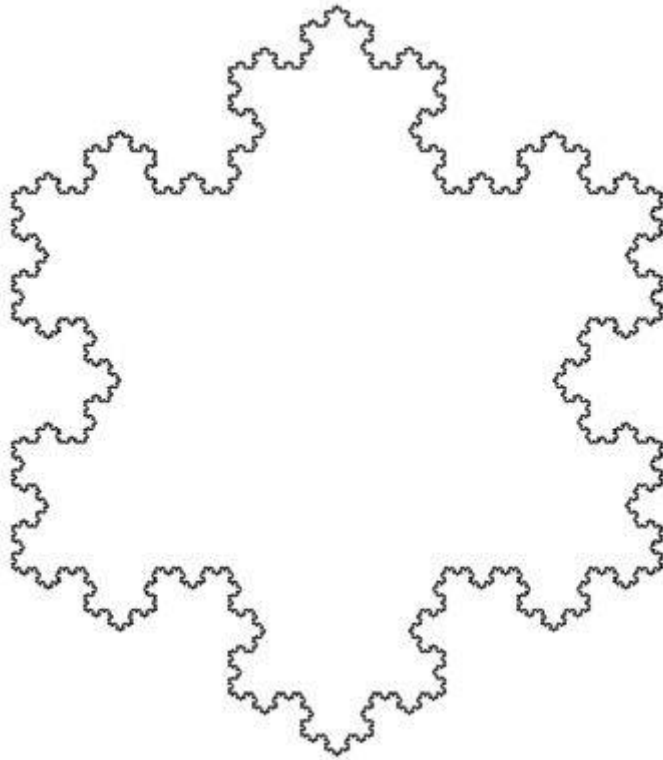
After doing this once the result should be a shape similar to the Star of David.



The Koch curve is in the limit approached as the above steps are followed over and over again.







The Koch curve has infinite length because each time the steps above are performed on each line segment of the figure its length increases by one third. The length at step  $n$  will therefore be  $(4/3)^n$ .

The area of the Koch snowflake is  $8/5$  that of the initial triangle, so an infinite perimeter encloses a finite area.

# What's Next?

Geometry for elementary school/What next?



# Notation

Geometry for elementary school		
Squares	Notation	

This chapter summarizes the notation used in the book. An effort was made to use common conventions in the notation. However, since many conventions exist the reader might see a different notation used in other books.

One who is interested seeing an example of past notation might be interested in Byrne's edition of Euclid's Elements (<http://www.sunsite.ubc.ca/DigitalMathArchive/Euclid/byrne.html>) . See for example the equilateral triangle construction (<http://www.sunsite.ubc.ca/DigitalMathArchive/Euclid/bookI/images/bookI-1.html>) .

## Point

A point will be named by a bolded English letter, as in the point **A**.

## Line segment

We will use the notation  $\overline{AB}$  for the line segment that starts at **A** and ends at **B**. Note that we don't care of the segment direction and therefore  $\overline{AB}$  is similar to  $\overline{BA}$  .

## Angle

We will use the notation  $\angle ABC$  for the angle going from the point **B**, the intersection point of the segments  $\overline{BA}$  and  $\overline{BC}$  .

## Triangle

A triangle whose vertices are **A**, **B** and **C** will be noted as  $\triangle ABC$ . Note that for the purpose of triangles' congruence, the order of vertices is important and  $\triangle ABC$  and  $\triangle BCA$  are not necessarily congruent.

## Circle

We use the notation  $\circ A, \overline{BC}$  for the circle whose center is the point **A** and its radius length equals that of the segment  $\overline{BC}$ .

Note that in other sources, such as Euclid's Elements (<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/defI15.html>), a circle is describe by any 3 points on its circumference, **ABC**.

The center, radius notation was chosen since it seems to be more suitable for constructions.

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